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Analysis of a discrete-time $Geo^X/G/1$ repairable queue with multiple vacations, feedback and p-entering discipline

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ABSTRACT

This paper considers a discrete-time $Geo^X/G/1$ repairable queue with server ' s multiple vacations, customer feedback and p-entering discipline. The arriving batch enters the system with probability p or is lost with probability 1-p during server vacations. The customer who has just been served returns to the queue with probability 1-q for another service or leaves permanently with probability q. The server may break down during service and does not continue to work until it is repaired. Using the z-transform and renewal process theory, the reliability indices of the transient and steady-state unavailability, the expected failure number during (0+, n+] and the steady-state failure frequency of server are studied. The transient structure of the server reliability indices is characterized. As a real-world application the reliability indices of the proxy server in a network access proxy system are analyzed numerically.

Keywords: repairable queue; batch arrival; multiple vacations; customer feedback; proxy server

1. Introduction

Discrete-time vacation queues have been widely studied over the past decades due to their wide applications in broadband integrated service digital networks (BISDN), asynchronous transfer mode (ATM) and computer telecommunication systems. For a comprehensive review of the main results, methods and applications, readers may refer to the papers [1-7], the books [8-9] and their references. Based on the fact that the sever of a queueing system is subject to unpredictable breakdowns and repairs when serving a customer, some researchers, such as Tang et al.[10], Liu and Gao[11], Lan and Tang[12,13], Kulshrestha et al.[14] and so on, analyzed the reliability of the server in some discrete -time repairable queueing systems. However, existing studies have mainly focused on the discrete-time repairable queues with a constant arrival rate. In fact, the customer arrival rate may have something to do with server states. For example, in a telecommunication network the arrival rate of a message may vary when the server is under a maintenance activity (e.g. virus scan). On the other hand, the feedback is a common phenomenon in real world, e.g., in telecommunication systems the messages that produce errors at the destination need to be sent again. In a call center a user may call again when their problem is not completely solved. Thus the reliability study of discrete-time repairable queues with vacations, feedback and variable arrival rate is not only significant for theoretical investigations but also valuable for practical applications.

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In this paper, we consider the reliability of the server for a discrete-time $Geo^X/G/1$ queue with vacations, feedback and p-entering discipline. The arriving batch enters the system with probability p or is lost with probability 1-p during server vacations. The customer who has just been served returns to the queue with probability 1-q for another service or leaves permanently with probability q. The server may break down during service and does not continue to work until it is repaired. Using the z-transform and renewal process theory, the server reliability indices, such as the transient and steady-state unavailability, the expected failure number during (0+, n+] and the steady-state failure frequency, are studied. The transient structure of the server reliability indices in this type of repairable queue is characterized.

The proposed queueing system can be used to model a network access proxy system (NAPS). In such a system, maintenance activities (vacations), such as virus scanning, are required to keep the proxy server operating properly. After completing one maintenance activity, the proxy server either performs another maintenance activity or begins serving waiting requests. A service request received with errors at the destination may be retransmitted (feedback). In addition, a busy server may stop working when unpredictable events occur, such as network congestion. The service interruption is repaired immediately, and once the interruption is recovered, the server resumes service. Because channel requests, grants, data transmissions, and receptions all occur in fixed time intervals, and because the batch-request arrival rate differs between the server maintenance period and the busy period, this system can be modeled as a discrete-time repairable queue with vacations, feedback, and a variable arrival rate. Moreover, for the NAPS described above, the server reliability indices obtained in Section 3 are useful for analyzing the effects of system parameters on proxy-server performance (see Section 4).

The remainder of this paper is organized as follows. Section 2 presents the queueing assumptions and several preliminaries. Section 3 derives the server reliability indices using the z-transform and renewal-point techniques and discusses several special cases. Section 4 provides numerical examples to validate the theoretical results for a network access proxy system. Section 5 concludes the paper.

2. Queueing Assumptions and Preliminaries

We consider the following discrete-time $Geo^X/G/1$ repairable queue with multiple vacations, Bernoulli feedback and p-entering discipline. The model is considered under the LAS-DA (late arrival system with delayed access) setup[9] , that is, the batch arrival and repair-start epochs occur in (n-,n), and the customer departure and repair-completion epochs take place in (n,n+). The various time epochs involved in our model can be viewed through a schematic diagram (see Figure 1). When the server is on guard the batch arrival intervals, $\{\tau i, i\geq 1\}$ are independent identically distributed (i.i.d) random variables generated by a geometric distribution. $P\{\tau i=j\}=\lambda(1-\lambda)^{j-1}, (j\geqslant 1, 0<\lambda<1)$. During server vacations the arriving batch enters the system with probability p or is lost with probability 1-p,$(0<p\leqslant 1)$. Each batch size X is a random variable obeying an an arbitrary distribution $P\{X=k\}=x_k$, $k\geqslant 1$ with finite mean E(X) and probability-generating function (PGF) $X(z)=\sum_{k=1}^{\infty} x_k z^k, |z|<1$.

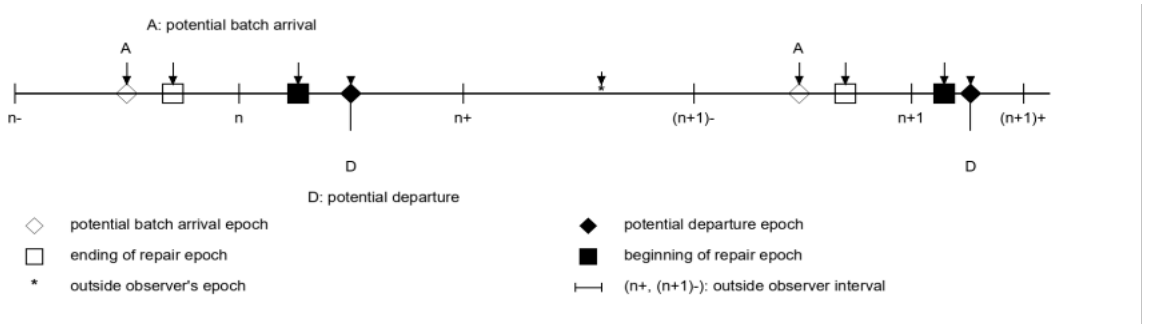


Figure 1 Various time epochs in a late arrival system with delayed access

Customers in different batches are served in FCFS order, and the service order within a batch is arbitrary. The server serves only one customer at a time. The service times, $\{\chi_n, n \geq 1\}$, are i.i.d random variables with distribution $g_j = P\{\chi_n = j\}, j \geq 1$ with PGF $G(z) = \sum_{j=1}^{\infty} g_j z^j, |z| < 1$ and mean service time $1/\mu$. The customer who has just been served returns to the queue with probability $1-q$ $(0 < q \leq 1)$ for another service or leaves permanently with probability q.

Whenever the system is empty the server takes multiple vacations. Upon returning from a vacation, the server immediately starts serving until the system becomes empty again when there is a waiting queue, or else the server leaves for another vacation. The server vacation times, $V_i, i \geq 1$, are i.i.d. random variables satisfying an arbitrary distribution $v_j = P\{V_i = j\}, j \geq 1$ with PGF $v(z) = \sum_{j=1}^{\infty} v_j z^j, |z| < 1$.

The server may break down during service. The broken server is repaired immediately. After repair, the server is as good as new and continues serving the customer whose service has not yet been completed. The service time for a customer is cumulative. Assume that the failure distribution of server is geometric with mean $1/\alpha, 0 < \alpha < 1$, and its repair time follows an an arbitrary distribution $r_j, j \geq 1$ with PGF $r(z) = \sum_{j=1}^{\infty} r_j z^j, |z| < 1$ and a mean repair time $1/\beta$.

At the initial time 0^+, the server is idle and waits for the first batch arrival when the customer number in the system $N(0^+) = 0$, or the server starts service when $N(0^+) > 0$. After the first busy period, the server begins to take multiple vacations. All random variables are mutually independent.

For later analysis, the following notation is used throughout this paper: $P(\Lambda)$ is the probability of event Λ. $N(n^+)$ denotes the system size at time n^+. $C_m^k = \frac{m!}{k!(m-k)!}, 0 \leq k \leq m$. $E(\xi)$ is the mean of random variable ξ. Also, we give some preliminaries as follows.

2.1. The batch arrival interval distribution during server vacations.

By the queueing assumptions, the batch arrival intervals during server vacations, denoted by $\tau_i^v, i = 1,2,\cdots$, i.i.d. random variables with geometric distribution $P\{\tau_i^v = j\} = p\lambda(1 - p\lambda)^{j-1}, j \geq 1$.

2.2. Generalized service time of a customer.

This is the length of time between the time when the service of a customer begins and the time when the service of this customer ends, which may include some repair times of the server due to server breakdowns while serving this customer.

Let $\tilde{\chi}$ denote the generalized service time of a customer, then from [2], the PGF of $\tilde{\chi}$ is given by

$$\tilde{G}(z) = \sum_{j=1}^{\infty} P(\tilde{\chi} = j) z^j = G[\alpha z r(z) + z(1 - \alpha)], \quad |z| < 1, \tag{2.1}$$

and the mean is

$$E(\tilde{\chi}) = \frac{\mathrm{d}\tilde{G}(z)}{\mathrm{d}z}|_{z=1} = \frac{\alpha + \beta}{\mu\beta}. \tag{2.2}$$

2.3. Total generalized service time of a customer before departure.

Denote by $\tilde{\chi}_n^i$, $\hat{\chi}_n$ and Λ_n the i th generalized service time, total generalized service time and total generalized service number of customer n before departure, respectively. Then according to queueing assumptions we get $P(\Lambda_n = k) = q(1 - q)^{k-1}, k \geq 1$, and $\hat{\chi}_n = \sum_{i=1}^{\Lambda_n} \tilde{\chi}_n^i, n \geq 1$ are i.i.d. random variables, where $\tilde{\chi}_n^i, i, n \geq 1$ are mutually independent and have the same distribution as $\tilde{\chi}$.

Furthermore, the distribution $\hat{g}_j$, PGF $\hat{G}(z)$ and mean $E(\hat{\chi})$ of total generalized service time $\hat{\chi}$ for one customer before departure are given, respectively, by

$$\hat{g}_j = P\left\{\sum_{i=1}^{\Lambda_n} \tilde{\chi}_n^i = j\right\} = \sum_{k=1}^{j} q\,(1-q)^{k-1} P\left\{\sum_{i=1}^{k} \tilde{\chi}_n^i = j\right\}, j \geq 1, \tag{2.3}$$

$$\hat{G}(z) = \sum_{j=1}^{\infty} \hat{g}_j\, z^j = \frac{q\tilde{G}(z)}{1-(1-q)\tilde{G}(z)}, |z| < 1, \tag{2.4}$$

$$E(\hat{\chi}) = \frac{\mathrm{d}\hat{G}(z)}{\mathrm{d}z}|_{z=1} = \frac{\alpha+\beta}{\mu q \beta}, \tag{2.5}$$

where $\tilde{G}(z)$ is given by (2.1).

2.4. Server busy period.

This is the length of time between the time when the server begins service and the time when the system becomes empty, which may contain some repair times of server due to server breakdowns during service.

Let b be the server busy period beginning with one customer and its PGF is denoted by $b(z), |z| < 1$. Based on a similar analysis in an M/G/1 vacation queue[15], we obtain $b(z)$ is the root of the equation $b(z) = \hat{G}[z - \lambda z + \lambda X(b(z))z]$, and the mean is given by

$$E(b) = \begin{cases} \dfrac{\rho}{\lambda E(X)(1-\rho)}, & \rho < 1, \\ \infty, & \rho \geq 1, \end{cases} \tag{2.6}$$

where $\rho = \lambda E(X)(\alpha+\beta)/(\mu q \beta)$ is the traffic intensity of the discussed queue.

Let $b^{<i>}$ represent the server busy period beginning with i customers, then $b^{<i>}$ can be decomposed as $b^{<i>} = \Omega_1 + \cdots + \Omega_i$, where $\Omega_1, \cdots, \Omega_i$ are mutually independent and follow the same distribution as b. Thus the PGF of $b^{<i>}$ is expressed as $b^i(z)$.

3. Reliability indices analysis of the server

In this section, we first discuss the conditional probability that time n+ is during server busy period, the unavailability and the mean number of failures during (0+, n+]. Next, we reveal the transient structure of serv er reliability indices. Finally, we present the server reliability indices for some special queues.

3.1. The conditional probability that the time n+ is during server busy period.

Let $\Pi_i(n^+) = P(\text{the time n^+ is during server busy period}|N(0^+) = i), i \geq 0$, and its PGF is $\pi_i(z) = \sum_{n=0}^{\infty} \Pi_i(n^+) z^n$, $|z| < 1$. We denote

$$s_k = \sum_{i=1}^{k} V_i, \quad l_k = \sum_{i=1}^{k} \tau_i^v, k \geq 1, \quad s_0 = l_0 = 0, \quad \sum_{m[i]=i}^{\infty} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_i=1}^{\infty}, \quad m[i] = m_1 + \ldots + m_i.$$

Denote by $b^{<k>}$ the server busy period beginning with k customers with PGF $b^k(z)$ (see subsection 2.4). Based on the fact that the end points of server busy period and server vacation are renewal points, and the assumption that the server takes no vacations when $N(0^+) = 0$, we can obtain

$$\Pi_0(n^+) = \sum_{k=1}^{\infty} x_k P\left(\tau_1 \le n^+ < \tau_1 + b^{<k>}\right) + \sum_{k=1}^{n} x_k \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} P\,(s_{j-1} < \tau_1^v \le s_j,\ l_i < s_j \le l_{i+1},\ n^+ \ge \tau_1 + b^{<k>} + s_j,\ \text{the time } n^+ \text{ is during server busy period}). \tag{3.1}$$

The first term of (3.1) means that there are k arrivals at the initiation point of the first busy period b_1 of server and the time n^+ locates in b_1. As the beginning epoch of b_1 is a renewal point, using renewal process theory, the first term of (3.1) is expressed as

$$\sum_{k=1}^{\infty} x_k \sum_{i=1}^{n} P\,(\tau_1 = i) P\left(b^{<k>} > (n-i)^+\right). \tag{3.2}$$

In the second term of (3.1), the "$s_{j-1} < \tau_1^v \le s_j,\ l_i < s_j \le l_{i+1}$" implies that the first batch arrives during the jth vacation period V_j and total arriving batch number in V_j is i. It means that at end point of V_j total arrival number equals $m[i]$. Since the end point of V_j is the beginning point of a server busy period, which is a renewal point, the second term of (3.1) is given by

$$\sum_{k=1}^{n} x_k \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{m[i]=i}^{\infty} x_{m_1}\ldots x_{m_i} \sum_{u=k+1}^{n}\sum_{t=j-1}^{n-u}\sum_{r=1}^{n-u-t} \Pi_{m[i]}\left((n-u-t-r)^+\right) \times P(\tau_1 + b^{<k>} = u) P(s_{j-1} = t) v_r C_r^i (p\lambda)^i (1-p\lambda)^{r-i} (1-p\lambda)^t. \tag{3.3}$$

Multiplying (3.1) by z^n and summing over n after substituting (3.2) and (3.3) into (3.1), we obtain

$$\pi_0(z) = \frac{\lambda z}{(1-z)(1-z+\lambda z)}[1 - X(b(z))] + \frac{\lambda z X(b(z))}{(1-z+\lambda z)[1-v(z-p\lambda z)]} \times \sum_{i=1}^{\infty}\sum_{m[i]=i}^{\infty} x_{m_1}\ldots x_{m_i}\pi_{m[j]}(z) \sum_{r=1}^{\infty} v_r\, z^r C_r^i (p\lambda)^i (1-p\lambda)^{r-i}, \tag{3.4}$$

where $b(z), X(z)$, and $v(z)$ are given by queueing assumptions.

For $i \ge 1$, $\Pi_i(n^+)$ shows that there are i customers at initial time 0^+. Based on the supposition in Section 2, the server busy period $b^{<i>}$ will begin at time 0^+. Similar to the analysis of $\Pi_0(n^+)$, we have

$$\begin{aligned}\Pi_i(n^+) =& P(n^+ < b^{<i>}) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} P\,(s_{k-1} < \tau_1^v \le s_k,\ l_j < s_k \le l_{j+1},\ n^+ \ge b^{<i>} + s_k,\\ & \text{the time } n^+ \text{ is during server busy period})\\ =& 1 - \sum_{k=i}^{n} P\,(b^{<i>} = k) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m[j]=j}^{\infty} x_{m_1}\ldots x_{m_j} \sum_{u=i}^{n}\sum_{t=k-1}^{n-u}\sum_{r=1}^{n-u-t} \Pi_{m[j]}\left((n-u-t-r)^+\right)\\ & \times P(b^{<i>} = u) P(s_{k-1} = t) v_r C_r^j (p\lambda)^j (1-p\lambda)^{r-j} (1-p\lambda)^t.\end{aligned} \tag{3.5}$$

Taking z-transform of (3.5), we get, for $i \ge 1$,

$$\pi_i(z) = \frac{1-b^i(z)}{1-z} + \frac{b^i(z)}{1-v(z-p\lambda z)} \sum_{j=1}^{\infty}\sum_{m[j]=j}^{\infty} x_{m_1}\ldots x_{m_j}\pi_{m[j]}(z) \sum_{r=1}^{\infty} v_r\, z^r C_r^j (p\lambda)^j (1-p\lambda)^{r-j}. \tag{3.6}$$

Solving (3.4) and (3.6) and using the following equation:

$$\sum_{j=1}^{\infty}\sum_{m[j]=j}^{\infty} x_{m_1}\ldots x_{m_j} b^{m[j]}(z)\sum_{r=1}^{\infty} v_r\, z^r C_r^j (p\lambda)^j (1-p\lambda)^{r-j} = v(z-p\lambda z+p\lambda z X(b(z))) - v(z-p\lambda z),$$

we obtain

$$\pi_0(z) = \frac{\lambda z}{(1-z)(1-z+\lambda z)}\left\{1-\frac{X(b(z))(1-v(z))}{1-v\left[z-p\lambda z\left(1-X(b(z))\right)\right]}\right\},\ |z|<1, \tag{3.7}$$

$$\pi_i(z) = \frac{1}{1-z}\left\{1-\frac{b^i(z)[1-v(z)]}{1-v\left[z-p\lambda z\left(1-X(b(z))\right)\right]}\right\},\ i\geq 1,\ |z|<1. \tag{3.8}$$

It follows from the limitation theory of z-transform (see [2]) that $\lim_{n\to\infty}\Pi_i(n^+) = \lim_{z\uparrow 1}(1-z)\pi_i(z)$. Applying (3.7), (3.8), L'Hospital's rule, and (2.6), for $\rho = \lambda E(X)(\alpha+\beta)/(q\mu\beta)$, the steady-state busy probability of server, denoted by Π, is

$$\Pi = \lim_{n\to\infty}\Pi_i(n^+) = \lim_{z\uparrow 1}(1-z)\pi_i(z) = \begin{cases}\dfrac{p\lambda E(X)(\alpha+\beta)}{\mu q\beta-(1-p)\lambda E(X)(\alpha+\beta)}, & \rho<1,\\ 1, & \rho\geq 1.\end{cases} \tag{3.9}$$

To analyze the unavailability and the number of failures during $(0^+, n^+]$ of server, we introduce a classical discrete-time reliability system consisting of a single unit. The system breaks down if and only if the unit fails. The failed unit is immediately repaired and, after repair, is as good as new. When the failed unit is repaired, the system resumes operation immediately. The failure distribution of the unit is geometric with mean $1/\alpha, 0<\alpha<1$, and its repair time follows an arbitrarily discrete distribution with the PGF $r(z)$ and a mean repair time $1/\beta$. Assume that the unit is new at the initial time 0^+, and the failure and repair of the unit are mutually independent. For $n\geq 0,\ |z|<1,$ let

$$U(n^+) = P(\text{the system is repaired at time } n^+),\ u(z) = \sum_{n=0}^{\infty} U(n^+)z^n, \tag{3.10}$$

$$F(n^+) = E(\text{the failure number of the system during } (0^+, n^+]),\ f(z) = \sum_{n=0}^{\infty} F(n^+)z^n, \tag{3.11}$$

then similar to the discussions in a continuing-time single-unit reliability system[16], we have

$$u(z) = \frac{z\alpha[1-r(z)]}{(1-z)\left[1-z+z\alpha(1-r(z))\right]},\quad f(z) = \frac{z\alpha}{(1-z)\left[1-z+z\alpha(1-r(z))\right]},|z|<1, \tag{3.12}$$

$$\lim_{n\to\infty}U(n^+) = \lim_{z\uparrow 1}(1-z)u(z) = \frac{\alpha}{\alpha+\beta},\quad \lim_{n\to\infty}\frac{F(n^+)}{n} = \lim_{z\uparrow 1}(1-z)^2 f(z) = \frac{\alpha\beta}{\alpha+\beta}. \tag{3.13}$$

3.2. The unavailability of server.

The unavailability of server at time n^+ denotes the probability that the server is repaired at time n^+. Let $U_i(n^+) = P(\text{the server is repaired at time } n^+|N(0^+)=i)$, and its PGF is $u_i(z) = \sum_{n=0}^{\infty} U_i(n^+)z^n,\ |z|<1$.

From the assumptions in Section 2, we know that the server is under repair at time n^+ if and only if the time n^+ is within a server busy period, and the server is repaired at time n^+. Since the starting points of server busy period are regeneration points, by the law of total probability and the theory of renewal process, similarly to the analysis of $\Pi_0(n^+)$ in (3.1), $U_0(n^+)$ is expressed as

$$\begin{aligned} U_0(n^+) = & \sum_{k=1}^{\infty} x_k P\big(\tau_1 \le n < \tau_1 + b^{<k>}, \text{ the server is repaired at time } n^+\big) \\ & + \sum_{k=1}^{n} x_k \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P\,(s_{j-1} < \tau_1^v \le s_j, l_i < s_j \le l_{i+1}, \ n^+ \ge \tau_1 + b^{<k>} + s_j, \\ & \text{the server is repaired at time } n^+) \\ = & \sum_{k=1}^{\infty} x_k \sum_{i=1}^{n} P\,(\tau_1 = i) A_k((n-i)^+) \\ & + \sum_{k=1}^{n} x_k \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{m[i]=i}^{\infty} x_{m_1} \ldots x_{m_i} \sum_{u=k+1}^{n} \sum_{t=j-1}^{n-u} \sum_{r=1}^{n-u-t} U_{m[i]}\,((n-u-t-r)^+) \\ & \times P(\tau_1 + b^{<k>} = u) P(s_{j-1} = t) v_r C_r^i (p\lambda)^i (1-p\lambda)^{r-i} (1-p\lambda)^t, \end{aligned} \tag{3.14}$$

where $A_k(n^+) = P\big(0 \le n^+ < b^{<k>}, \text{ the server is repaired at time } n^+\big)$, $k \ge 1$.

Similarly, $U_i(n^+)$ is given as follows:

$$\begin{aligned} U_i(n^+) = A_i(n^+) + & \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m[j]=j}^{\infty} x_{m_1} \ldots x_{m_j} \sum_{u=i}^{n} \sum_{t=k-1}^{n-u} \sum_{r=1}^{n-u-r} U_{m[j]}\,((n-u-t-r)^+) \\ & \times P(b^{<i>} = u) P(s_{k-1} = t) v_r C_r^j (p\lambda)^j (1-p\lambda)^{r-j} (1-p\lambda)^t, \quad i \ge 1. \end{aligned} \tag{3.15}$$

By considering the server during busy period as a discrete-time single-unit reliability system and applying the law of total probability, we obtain

$$\begin{aligned} U(n^+) &= P\big(\text{the system is repaired at time } n^+, b^{<k>} > n^+\big) \\ &\quad + P\big(\text{the system is repaired at time } n^+, b^{<k>} \le n^+\big) \\ &= A_k(n^+) + \sum_{j=k}^{n} P\,\big(b^{<k>} = j\big) U((n-j)^+), \end{aligned} \tag{3.16}$$

which gives rise to

$$\sum_{n=0}^{\infty} A_k\,(n^+) z^n = u(z)\big[1 - b^k(z)\big], \ \ k \ge 1, \ \ |z| < 1, \tag{3.17}$$

where $U(n^+)$ and $u(z)$ are given by (3.10) and (3.12), respectively.

Taking z-transforms on (3.11) and (3.12) and using (3.17) yield

$$\begin{aligned} u_0(z) = & \frac{\lambda z u(z)}{1 - z + \lambda z}[1 - X(b(z))] + \frac{\lambda z X(b(z))}{(1 - z + \lambda z)[1 - v(z - p\lambda z)]} \\ & \times \sum_{i=1}^{\infty} \sum_{m[i]=i}^{\infty} x_{m_1} \ldots x_{m_i} u_{m[i]}(z) \sum_{r=1}^{\infty} v_r\, z^r C_r^i (p\lambda)^i (1-p\lambda)^{r-i}, \end{aligned} \tag{3.18}$$

$$u_i(z) = u(z)[1 - b^i(z)] + \frac{b^i(z)}{1 - v(z - p\lambda z)} \times \sum_{j=1}^{\infty} \sum_{m[j]=j}^{\infty} x_{m_1} \ldots x_{m_j} u_{m[j]}(z) \sum_{r=1}^{\infty} v_r z^r C_r^j (p\lambda)^j (1 - p\lambda)^{r-j}, \; i \geq 1. \quad (3.19)$$

Solving (3.18) and (3.19) and using (3.7) and (3.8), we get

$$u_i(z) = u(z)[(1 - z)\pi_i(z)], i \geq 0, |z| < 1. \quad (3.20)$$

Therefore, according to the limiting theorem of z-transform, (3.13) and (3.9), for $\rho = \lambda E(X)(\alpha + \beta)/(\mu q \beta)$, the steady-state unavailability of server, denoted by U, is

$$U = \lim_{n \to \infty} U_i(n^+) = \lim_{z \uparrow 1} (1 - z) u_i(z) = \begin{cases} \dfrac{p\lambda E(X)\alpha}{\mu q \beta - (1 - p)\lambda E(X)(\alpha + \beta)}, & \rho < 1, \\ \dfrac{\alpha}{\alpha + \beta}, & \rho \geq 1. \end{cases} \quad (3.21)$$

3.3. The mean failure number of server during (0+, n+]

Let $F_i(n^+) = E(\text{the failure number of server during } (0^+, n^+] | N(0^+) = i), i \geq 0$, and its PGF is $f_i(z) = \sum_{n=0}^{\infty} F_i(n^+) z^n, \; |z| < 1$. Also, for $k \geq 1$, let

$$W_k(n^+) = E\left(n^+ < b^{<k>}, \text{ the failure number of server during } (0^+, n^+]\right), \quad (3.22)$$

$$T_k(n^+) = E\left(b^{<k>} \leq n^+, \text{ the failure number of server during } \left(0^+, b^{<k>}\right]\right), \quad (3.23)$$

then by conducting a similar analysis in (3.16) and (3.17) above on $F(n^+)$, we obtain

$$W_k(n^+) + T_k(n^+) = F(n^+) - \sum_{j=k}^{n} P\left(b^{<k>} = j\right) F((n - j)^+), \; k \geq 1, \quad (3.24)$$

$$\sum_{n=0}^{\infty} [W_k(n^+) + T_k(n^+)] z^n = f(z)\left[1 - b^k(z)\right], \; k \geq 1, \; |z| < 1, \quad (3.25)$$

where $F(n^+)$ and $f(z)$ are given by (3.11) and (3.12), respectively.

For $F_0(n^+)$, using the law of total probability yields

$$\begin{aligned} F_0(n^+) = \sum_{k=1}^{\infty} x_k \{ & E\left(\tau_1 \leq n^+ < \tau_1 + b^{<k>}, \text{the failure number of server during} (\tau_1, n^+]\right) \\ & + E\left(n^+ \geq \tau_1 + b^{<k>}, \text{the failure number of server during} \left(\tau_1, \tau_1 + b^{<k>}\right]\right)\} \\ & + \sum_{k=1}^{n} x_k \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} E\left(n^+ \geq \tau_1 + b^{<k>} + s_j, \; s_{j-1} < \tau_1^v \leq s_j, \; l_i < s_j \leq l_{i+1}, \right. \\ & \left. \text{the failure number of server during} \left(\tau_1 + b^{<k>} + s_j, n^+\right]\right) \end{aligned} \quad (3.26)$$

Noting that the ending time of the first arrival interval τ_1 is the beginning point of the first server busy period, which is a regeneration point, by using renewal process theory, (3.22) and (3.23), the first term of (3.26) is expressed as

$$\sum_{k=1}^{\infty} x_k \sum_{i=1}^{n} [W_k((n - i)^+) + T_k\left((n - i)^+\right)] P(\tau_1 = i). \quad (3.27)$$

Similar to the analysis of (3.3), the second term of (3.26) is given by

$$\sum_{k=1}^{n} x_k \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{m[i]=i}^{\infty} x_{m_1} \ldots x_{m_i} \sum_{u=k+1}^{n} \sum_{t=j-1}^{n-u} \sum_{r=1}^{n-u-t} F_{m[i]}((n-u-t-r)^+) \times P(\tau_1 + b^{<k>} = u) P(s_{j-1} = t) v_r C_r^i (p\lambda)^i (1-p\lambda)^{r-i} (1-p\lambda)^t, \tag{3.28}$$

In the same way, we have

$$F_i(n^+) = W_i(n^+) + T_i(n^+) + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m[j]=j}^{\infty} x_{m_1} \ldots x_{m_j} \sum_{u=i}^{n} \sum_{t=j-1}^{n-u} \sum_{r=1}^{n-u-t} F_{m[j]}((n-u-t-r)^+) \times P(b^{<i>} = u) P(s_{k-1} = t) v_r C_r^j (p\lambda)^j (1-p\lambda)^{r-j} (1-p\lambda)^t, i \geq 1. \tag{3.29}$$

Taking z-transforms on (3.26)-(3.29), and applying (3.7), (3.8) and (3.12), we obtain

$$f_i(z) = f(z)[(1-z)\pi_i(z)], \quad i \geq 0. \tag{3.30}$$

Further, by the limiting theorem for the z-transform, (3.9) and (3.13), for $\rho = \lambda E(X)(\alpha+\beta)/(\mu q \beta)$, the steady-state failure frequency of server, that is, in steady-state, the server failure occurrence rate, denoted by F, is

$$F = \lim_{n\to\infty} \frac{F_i(n^+)}{n} = \lim_{z\uparrow 1} (1-z)^2 f_i(z) = \begin{cases} \dfrac{p\lambda E(X)\alpha\beta}{\mu q\beta - (1-p)\lambda E(X)(\alpha+\beta)}, & \rho < 1, \\ \dfrac{\alpha\beta}{\alpha+\beta}, & \rho \geq 1. \end{cases} \tag{3.31}$$

Remark 3.1. (3.20) and (3.30) reveal the transient structure of server reliability indices. That is, analyzing $U_i(n^+)$ and $F_i(n^+)$ is turned into analyzing $U(n^+)$ and $F(n^+)$ in the classical discrete-time single-unit reliability system, and $\Pi_i(n^+)$ presented in this paper. Since $u(z)$ and $f(z)$ in the classical discrete-time single-unit system are known (see (3.12)), we only need to analyze $\Pi_i(n^+)$ for $U_i(n^+)$ and $F_i(n^+)$. It is clear that (3.20) and (3.30) greatly reduces the analytical difficulty of $U_i(n^+)$ and $F_i(n^+)$.

Remark 3.2. It is observed from (3.9), (3.21) and (3.31) that Π, U and F are independent of the initial state $N(0^+) = i$, $i \geq 0$, and server vacations, which is consistent with intuition.

Remark 3.3 (special cases)

(1) If $q = 1$ and $P(X = 1) = 1$, then our model becomes a discrete-time Geo/G/1 queue with multiple vacations, a repairable server and variable arrival rate. In this case, for $\rho = \lambda(\alpha+\beta)/(\mu\beta)$, then

$$\Pi = \begin{cases} \dfrac{p\lambda(\alpha+\beta)}{\mu\beta - (1-p)\lambda(\alpha+\beta)}, & \rho < 1, \\ 1, & \rho \geq 1, \end{cases} \quad U = \begin{cases} \dfrac{p\lambda\alpha}{\mu\beta - (1-p)\lambda(\alpha+\beta)}, & \rho < 1, \\ \dfrac{\alpha}{\alpha+\beta}, & \rho \geq 1, \end{cases}$$

$$F = \begin{cases} \dfrac{p\lambda\alpha\beta}{\mu\beta - (1-p)\lambda(\alpha+\beta)}, & \rho < 1, \\ \dfrac{\alpha\beta}{\alpha+\beta}, & \rho \geq 1. \end{cases}$$

(2) If $p = 1, P(X = 1) = 1$, and $P(V_i = 0) = 1, i \geq 1$, then our model becomes a discrete-time Geo/G/1 queue with Bernoulli feedback and a repairable server. In this case, for $\rho = \lambda(\alpha + \beta)/(\mu q\beta)$, then

$$\Pi = \begin{cases} \lambda(\alpha+\beta)/(\mu q\beta), & \rho < 1 \\ 1, & \rho \geq 1, \end{cases} \quad U = \begin{cases} \lambda\alpha/(\mu q\beta), & \rho < 1 \\ \dfrac{\alpha}{\alpha+\beta}, & \rho \geq 1, \end{cases} \quad F = \begin{cases} \lambda\alpha/(\mu q), & \rho < 1 \\ \dfrac{\alpha\beta}{\alpha+\beta}, & \rho \geq 1. \end{cases}$$

(3) If $p = 1, q = 1$ and $P(V_i = 0) = 1, i \geq 1$, then our model becomes a discrete-time Geo ^{X}/G/1 queue with a repairable server. In this case, for $\rho = \lambda E(X)(\alpha + \beta)/(\mu\beta)$, then

$$\Pi = \begin{cases} \lambda E(X)(\alpha+\beta)/(\mu\beta), & \rho < 1 \\ 1, & \rho \geq 1, \end{cases} \quad U = \begin{cases} \lambda E(X)\alpha/(\mu\beta), & \rho < 1 \\ \dfrac{\alpha}{\alpha+\beta}, & \rho \geq 1, \end{cases} \quad F = \begin{cases} \lambda E(X)\alpha/\mu, & \rho < 1 \\ \dfrac{\alpha\beta}{\alpha+\beta}, & \rho \geq 1. \end{cases}$$

These results are very similar to those for the continuous-time counterpart [17].

4. Numerical Examples

As a real-world application, we numerically analyze the reliability indices of the proxy server in a network access proxy system introduced in Section 1. Since channel requests, grants, data transmissions, and receptions in such a system all proceed in fixed time intervals, this system can be considered as a discrete-time one. It is assumed that service requests sent in batches by users are a Bernoulli process with rate λ. Each batch request size follows a geometric distribution with mean $1/\gamma$. During maintenance activities batches of service requests sent by users are assumed to wait for service with probability p or be lost with probability $1 - p$. The service time of each request is geometric with mean $1/\mu = 2$. A service request received with errors at the destination may be retransmitted (feedback) with probability $1 - q$. Also, we assume that unpredictable events inducing service interruption occur according to a Bernoulli process with rate α and service interruption is immediately recovered. The recovery time obeys a geometric distribution with parameter $\beta = 0.5$. Once the interruption is recovered the server will continuously start its service.

From (3.9), (3.21) and (3.31), we obtain values of the server reliability indices (see Tables 1–5). For analysis, the traffic intensity ρ of the network system is also considered in Tables 1-5.

Table 1 The reliability indices of the proxy server for different batch request arrival rate λ during server busy period $(p = 0.6, q = 0.85, \gamma = 0.5, \alpha = 0.35)$

λ	ρ	Π	U	F
0.05	0.4000	0.2857	0.1176	0.0588
0.10	0.8000	0.7059	0.2907	0.1453
0.15	1.2000	1.0000	0.4118	0.2059
0.20	1.6000	1.0000	0.4118	0.2059
0.25	2.0000	1.0000	0.4118	0.2059
0.30	2.4000	1.0000	0.4118	0.2059

Table 2 The reliability indices of the proxy server for different batch request entering probability p during server maintenance activities $(\lambda = 0.1, q = 0.85, \gamma = 0.5, \alpha = 0.35)$

p	ρ	Π	U	F
0.2	0.8000	0.4444	0.1830	0.0915
0.3	0.8000	0.5455	0.2246	0.1123
0.4	0.8000	0.6154	0.2534	0.1267
0.5	0.8000	0.6667	0.2745	0.1373
0.6	0.8000	0.7059	0.2907	0.1453
0.7	0.8000	0.7368	0.3034	0.1517

Table 3 The reliability indices of the proxy server for different request retransmission probability 1-q $(\lambda = 0.1, p = 0.6, \gamma = 0.5, \alpha = 0.35)$

q	ρ	Π	U	F
0.4	1.7000	1.0000	0.4118	0.2059
0.5	1.3600	1.0000	0.4118	0.2059
0.6	1.1333	1.0000	0.4118	0.2059
0.7	0.9714	0.9533	0.3925	0.1963
0.8	0.8500	0.7727	0.3182	0.1591
0.9	0.7556	0.6497	0.2675	0.1338

Table 4 The reliability indices of the proxy server for different parameter value γ of each batch request size $(\lambda = 0.1, p = 0.6, q = 0.85, \alpha = 0.35)$

γ	ρ	Π	U	F
0.3	1.3333	1.0000	0.4118	0.2059
0.4	1.0000	1.0000	0.4118	0.2059
0.5	0.8000	0.7059	0.2907	0.1453
0.6	0.6667	0.5455	0.2246	0.1123
0.7	0.5714	0.4444	0.1830	0.0915
0.8	0.5000	0.3750	0.1544	0.0772

Table 5 The reliability indices of the proxy server for different occurrence rate α of unpredictable events $(\lambda = 0.1, p = 0.6, q = 0.85, \gamma = 0.5)$

α	ρ	Π	U	F
0.25	0.7059	0.5902	0.1967	0.0984
0.35	0.8000	0.7059	0.2907	0.1453
0.45	0.8941	0.8352	0.3956	0.1978
0.55	0.9882	0.9805	0.5136	0.2568
0.65	1.0824	1.0000	0.5652	0.2826
0.75	1.1765	1.0000	0.6000	0.3000

Table 1 illustrates the effect of batch request arrival rate λ during server busy period on server performance. It is observed that as λ increases three server reliability indices increase initially and then remain unchanged. This is because the network system changes from stable to unstable. Table 2 shows the effects of batch request entering probability p during server maintenance activities on server performance. We see that p does not affect the traffic intensity and the system is invariably stable, but Π, U and F all increase with increasing values of p, which is as expected. Table 3 shows that a small request retransmission probability $1 - q$ leads to smaller values of all four indices. The effects of the parameter value γ of each batch request size onserver reliability indices are shown in Table 4. It is clear that the influence of γ on four indices is contrary to that of λ. From Table 5 we know that when the occurrence rate of unpredictable events α increases, the server reliability indices all increase monotonically. What is more, when the value α is greater than or equal to 0.65, the network system is unstable and the server remains continuously busy. The trends illustrated by Tables 1-5 are consistent with intuitive expectations.

5. Conclusion

In this paper, a discrete-time batch arrival repairable queue with multiple vacations, feedback and p-entering discipline is considered. Some reliability performance indices of the server, such as the busy probability, unavailability, failure frequency, are studied. The transient structure of the server reliability indices is revealed, which is useful for reducing analysis of server reliability indices. As special cases we present reliability results for several discrete-time repairable queues. For illustration the reliability indices of the proxy server in a network proxy system are obtained numerically. Furthermore, the effects of system parameters on proxy server reliability indices are also analyzed for practical applications. In future study, we will investigate the reliability problems and real-world applications of more complex discrete-time repairable queueing systems.

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Conflict of Interest

For the publication of this paper, the authors declare that there is no conflict of interest.

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